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Remaining Useful Life Estimation Based on a Nonlinear Diffusion Degradation Process
Xiao-Sheng Si, Wenbin Wang, Chang-Hua Hu, Dong-Hua Zhou, Senior Member, IEEE, and Michael G. Pecht, Fellow, IEEE

Abstract—Remaining useful life estimation is central to the prognostics and health management of systems, particularly for safety-critical systems, and systems that are very expensive. We present a non-linear model to estimate the remaining useful life of a system based on monitored degradation signals. A diffusion process with a nonlinear drift coefficient with a constant threshold was transformed to a linear model with a variable threshold to characterize the dynamics and nonlinearity of the degradation process. This new diffusion process contrasts sharply with existing models that use a linear drift, and also with models that use a linear drift based on transformed data that were originally nonlinear. Both existing models are based on a constant threshold. To estimate the remaining useful life, an analytical approximation of the first hitting time of the diffusion process across a threshold level is obtained in a closed form by a time-space transformation under a mild assumption. The unknown parameters in the established model are estimated using the maximum likelihood estimation approach, and goodness of fit measures are applied. The usefulness of the proposed model is demonstrated by several real-world examples. The results reveal that considering nonlinearity in the degradation process can significantly improve the accuracy of remaining useful life estimation.

Index Terms—Brownian motion, degradation, diffusion process, first hitting time, maximum likelihood, nonlinear drift, remaining useful life.

ACRONYMS

AIC Akaike information criterion

BM Brownian motion
CDF Cumulative probability function
FHT First hitting time
FPK Fokker-Planck-Kolmogorov
INS Inertial navigation system
MLE Maximum likelihood estimation
MSE Mean squared error
MTTF Mean time to failure
PDF Probability density function
RUL Remaining useful life

NOTATION

\( B(\cdot) \) Standard BM
\( M_k \) Degradation model \( k \)
\( L_t \) Random variable of the RUL at time \( F \)
\( T \) Lifetime
\( X(t) \) Degradation at time \( t \)
\( w \) Threshold level of \( X(t) \)
\( S_B(t) \) Threshold of \( B(t) \)
\( \mu(t; \theta) \) Drift coefficient of \( X(t) \)
\( \sigma_B \) Diffusion coefficient of \( X(t) \)
\( p_X(t) \) PDF of the HFT of the \( X(t) \) crossing \( w \)
\( p_B(t) \) PDF of the HFT of the \( B(t) \) crossing \( S_B(t) \)
\( f_{\tau|X_k} \) PDF of the FHT conditional on \( X_k \) and \( \theta \) at time \( t \)
\( f_{\tau|L_t} \) PDF of the RUL under model \( M_k \) at time \( t \)
\( \theta \) Parameter vector in \( \mu(t; \theta) \)
\( a \) Random effect in \( \mu(t; \theta) \) following \( N(\mu_a, \Sigma_a) \)
\( b \) Fixed parameters in \( \mu(t; \theta) \)

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I. INTRODUCTION

Because of limited natural resources, considerably increased safety and environmental concerns, and the drive to reduce operating costs, critical assets need to be managed over their entire life cycle, from design, manufacture, sale, and operation, to their end of life, to optimize life cycle management and reduce negative impacts on the environment [1], [2]. For safety-critical equipment, such as aviation control systems and nuclear power generators, the accurate, early estimation of failure is critical to avoid catastrophic events that may cause severe damage to equipment, loss of human lives, and environmental disasters. A recent example was the explosion of the Deepwater Horizon oil well in the Gulf of Mexico, which resulted in the loss of lives, harm to the environment, and adverse impact on citizens. There were indications that this disaster indirectly resulted from overuse of the system due to strong financial pressure. But the direct cause of impact was the failure of the ‘last line of defence,’ the so-called blowout preventer. The application of prognostics and health management (PHM) could have addressed this problem [3].

PHM is a methodology that permits the assessment of the reliability of a system under its actual application conditions, and exercises necessary management actions. Prognostics is a key step in PHM. Prognostics utilizes in-situ monitoring and analysis to assess system degradation, and determine the remaining useful life (RUL) of an asset. The RUL of an asset is defined as the length of time from the present time to the end of useful life. The need for RUL estimation is obvious because it relates to a frequently asked question in industry, which is how long a monitored asset can survive based on the available information. Based on the RUL estimation, appropriate actions can be planned. Especially for critical equipment, such as aircraft engines, or inertial navigation platforms used in aerospace and weapon systems, determining if and when to take equipment out of service is important from both a cost-effective point of view and a safety point of view.

It is critically important to assess the RUL of an asset while it is in use, as this information impacts the planning of maintenance activities, the supply chain, replenishment of the inventory system, operational performance, and profitability of the owner of an asset [4]–[12]. RUL estimation has also played an important role in the management of product reuse and recycling, which has an impact on human life, energy consumption, raw material use, pollution, and landfills [2], [13], [14]. The reused products must have sufficiently long lives to be able to be reused.

In the early 1980s, Derman et al. [15] demonstrated the usefulness of lifetime distributions in the context of system life extension. Traditional failure-time analysis methods for estimating component lifetimes were heavily dependent on the time-to-failure data, or lifetime data [16], [17]. However, some critical, valuable systems are not allowed to run to failure, or the tests to obtain failure information are very expensive. Therefore, lifetime data are often hard to obtain. In such cases, degradation data can be used as an alternate resource for lifetime analysis from an economical and practical viewpoint [18]–[23]. In many situations, such as the drift degradation of an inertial navigation system (INS) used in the aerospace industry, it is natural to view the failure event of interest as the result of a stochastic degradation process crossing a threshold level, i.e., to model the hitting time of the degradation as a time-dependent stochastic process. Singpurwalla [24], Cox [24], and Aalen and Gjessing [26] all advocated the development, adoption, and exploration of stochastic dynamic models in theory and practice for reliability estimation. On the other hand, dynamic environments induce changes in the physics of failure. Hence a stochastic process approach to model degradation provides flexibility with respect to describing the failure-generating mechanisms and characteristics of the operating environment. A key idea behind this approach is that the lifetime can be defined as the first hitting time (FHT) of the degradation process reaching a failure threshold, and the probability density function (PDF) of the failure time is then modeled as the PDF of the FHT of the underlying stochastic process to mimic the true failure time. Lee and Whitmore [27] have given a comprehensive review of a variety of FHT models, and have discussed their potential. Even when the degradation process is observed to be at or above the threshold level it does not necessarily mean that the system has actually broken down. Such a threshold level is just a pre-set warning level that is set as a boundary for the observed degradation for a variety of reasons.

Among stochastic process–based models, diffusion processes are types of random processes capable of describing random degradation [28], [29]. Brownian motion (BM) with a linear drift is a special diffusion process that has become very popular for degradation modeling in recent years. One of the most important advantages of BM with a linear drift is that the distribution of the FHT can be formulated analytically. This formulation is known as the inverse Gaussian distribution; it has many merits, and has been applied in reliability and lifetime analysis since the 1970s [30]. A fundamental problem related to BM with a linear drift is that it can only describe a linearly drifted diffusion process. However, nonlinearity exists extensively in practice, and the linear model cannot trace the dynamics of such a degradation process. It is more likely that degradation may accelerate at a later stage of life. Therefore, for a model to be realistic, it should incorporate some nonlinear structures. However, this issue has not been well documented in the literature.

Some nonlinear processes can be approximated to be nearly linear by some kind of transformation on the degradation data, such as a log-transformation [31], [32], or a time-scale transformation [33]–[35]. But these processes are limited to the cases in which such transformations exist, and not many nonlinear processes can be transformed in these ways. There is also an implicit assumption used in the above transformation that the random part of the transformed process is still BM, which may not always be the case. This gap leads to the primary purpose
of this paper, to provide a useful model that can model general nonlinear degradation directly without the use of the transformations mentioned above. This ability would be particularly useful in cases where the nonlinear degradation process cannot be transformed into a linear degradation by manipulating the data. From a mathematical and application point of view, the difficulties encountered are mainly two kinds. First, as for a nonlinear-drifted diffusion process, the distribution of FHT is related to solving the Fokker-Planck-Kolmogorov (FPK) equation with boundary constraints [36]. For a nonlinear case, this task is rather difficult, and at times impossible. Second, the available closed forms of the PDFs of the FHT for nonlinear-drifted diffusion processes are very limited in special cases [37], [38]. Simulations can be used to deal with nonlinear cases, and a numerical solution approximation method was developed by Nardo et al. [39]. However, neither can provide a closed form of the PDF of the FHT; and both are subject to long computation times, and require a lot of memory. However, in the context of PHM, it is necessary and valuable to derive and evaluate the PDF of FHT with a closed form from the degradation process, and provide rapid online estimation for reliability analysis and maintenance scheduling.

From the above analysis, it is clear that nonlinear degradation processes have not been studied thoroughly. To formulate the distribution of RUL, we first transform the problem from calculating the FHT distribution of the nonlinear diffusion process crossing a constant threshold into a standard BM crossing a time-dependent boundary. This transformation is achieved by a well-known time-space transformation developed by Ricciardi [40]. Such a transformation is not applied to the degradation data, but rather it is applied to the model, which differs from the transformations in [31], [33]–[35].

Because there is no closed form for such a distribution, an analytical approximation of the distribution of the FHT of the transformed process is obtained in a closed form under a mild assumption. This contribution is important, but has not been reported before. The unknown parameters in the degradation model are obtained using the maximum likelihood estimation (MLE) method, and we use two goodness-of-fit measures to compare the model fit. We demonstrate, using some real-world data sets, the usefulness of the proposed model, and the necessity of incorporating a nonlinear structure into the degradation model, which has significantly improved the accuracy of RUL estimation.

The remaining parts of the paper are organized as follows. Section II reviews the related literature with an emphasis on RUL modeling methods based on BM with drift. Section III presents two motivating examples, and the modeling principle of our degradation model. Section IV proposes the main theoretical results for calculating the PDF of the FHT, and the procedure for parameter estimation. Three practical examples are presented to verify the proposed model in Section V. Section VI concludes the paper with a discussion.

II. LITERATURE REVIEW

There are many methods for estimating RUL [3], [9], [41]. The methodologies are largely classified into physics-of-failure, machine learning, and statistics-based methods. Si et al. [42] presented a review on statistics-based methods for RUL estimation. In statistics-based methods, there are models relying on the availability of actual failure data, as well as models needing a threshold for degradation to define the failure. BM with drift belongs to the latter.

BM with drift was originally used to depict the random walk of small particles in fluids, and air in physics. The walk has a trend, so it is appropriate for modeling a dynamic process with an increasing or decreasing trend. The random term is Gaussian distributed, so it is not a monotonic process. Due to its clarity in concept and its similarity to physical degradation processes, it has been widely applied in degradation modeling, and further extended to RUL estimation. BM with drift is a type of stochastic process with Gaussian noise. The variance of the noise is a function of time, and therefore the cumulative degradation is infinitely divisible, as required by any physical degradation process [43]. This result implies that the degradation described by BM with drift is free from the constraint of the sampling frequency and intervals. This is, to our understanding, a necessary property for a stochastic process to be able to model a physical degradation process. The only other stochastic process to possess such a property is the Gamma process [43].

Tseng et al. [44] used BM with drift to determine the lifetime for the light intensity of LED lamps of contact image scanners. As an extension, Tseng and Peng [45] proposed an integrated Wiener process to model the cumulative degradation path of a product’s quality characteristics. Lee and Tang [46] handled the failure time prediction problem based on BM with drift under a time-censored degradation test, in which a modified expectation and maximization algorithm was used to estimate the mean failure time. A recent extension of the intermediate data for lifetime estimation can be found in [47]. Padgett and Tomlinson [48] applied BM with drift to model accelerated degradation data, and make an inference about the lifetime under a typical operating environment. Park and Padgett [49], [50] applied BM with drift to model the initial damage under accelerated testing, and infer the lifetime. Park and Padgett [32], and Gebrael et al. [9] used a log-transformation to transform the exponential path into a linear path, and then used BM with drift to conduct lifetime estimation. Joseph and Yu [51] assumed that there existed some transformations that could transform a nonlinear degradation process into a linear process, and then they used BM with drift for degradation modeling and reliability improvement. Balka et al. [52] reviewed some methods of cure models based on FHT for BM with drift. Though their review focused mainly on cure rate modeling in the biostatistics field, the principle is the same as our RUL modeling. Further, Peng and Tseng [53] incorporated the random effect in the drift coefficient, and measurement errors in BM, with a drift-based degradation process for lifetime assessment. In a recent paper, Wang et al. [54] considered BM with drift using an adaptive random drift coefficient for RUL estimation. From above, we observe that, though many extensions of RUL estimation methods using BM with drift have appeared, almost none of the previous models have explicitly considered the nonlinearity of the underlying degradation process without manipulating the degradation data. However, there is the exception of the work of Tseng and Peng [55]. In [55], Tseng and Peng addressed this...
problem by using a stochastic differential equation for degradation modeling of LEDs with the restrictive assumption that the ratio between the expectation and the variance of the derivative of the degradation process was a constant. We note that such an assumption is difficult to justify in practice. Therefore, we can state that there are issues and challenges remaining to be solved.

In summary, the literature on RUL estimation based on BM with drift has treated only simple structures (linear or linearized by some kind of transformation on the degradation data, as discussed earlier). Nonlinearity has not been directly considered in the literature. In the next section, we will give two practical examples to show the necessity of incorporating a nonlinear structure into a degradation process model. Because of the nonlinearity introduced, the exact closed form of the PDF of the RUL does not exist, and we develop an approximate closed form under a mild assumption. To our knowledge, this paper is the first to consider such nonlinearity with a closed form for RUL estimation based on BM with drift in the context of degradation modeling without using restrictive assumptions and data transformations. We start first with the motivating examples, and discuss the RUL modeling principle.

### III. Motivating Examples, and Our RUL Modeling Principle

#### Example 1: Drift Coefficient in an Inertial Navigation Platform

The inertial platform is a key component in the inertial navigation systems of weapon systems, and space equipment. Its operating state has a direct influence on navigation precision. The sensors fixed in an inertial platform mainly include three gyros, and three accelerometers, which measure angular velocity, and linear acceleration, respectively. Statistical analysis shows that almost 70% of the failures of inertial platforms result from gyroscopic drift. In our case, the gyro fixed on an inertial platform is a mechanical structure having two degrees of freedom from the driver and sense axis (see [56], [57] for a general description of inertial navigation platforms and gyros).

When the inertial platform is operating, the wheels of gyros rotate at very high speeds, which can lead to rotation axis wear, and finally result in the gyros’ drift. With the accumulation of wear, the drift increases, and finally results in the failure of gyros. As such, the drift of gyros is used as a performance indicator to evaluate the health condition of an inertial platform.

In our study, we only take the drift degradation measurement along the sense axis for illustrative purposes, because this variable plays a dominant role in the assessment of gyro degradation. The degradation data, including 5 tested items, and 9 measurements for each item, were obtained from inertial platforms’ precision tests, where the conditions were similar to a field setting. The data are shown in Table I, and are also shown graphically in Fig. 1. We can clearly observe nonlinear characteristics in drift degradation in Fig. 1.

#### Example 2: Fatigue-Crack Length in 2017-T4 Aluminum Alloy

2017-T4 aluminum alloy is a common metallic material used in aircraft [58]. The integrity of such a material is assessed by the length of fatigue cracks. When the length of a fatigue crack is equal to or more than a predefined threshold level of 6 mm, a structure made of this material is considered to be in a very critical state, and is defined as failed. The obtained data represent crack length propagation in 4 test specimens of 2017-T4 aluminum alloy under a stress level of 200 MPa, which simulates actual operating conditions. For each item, the fatigue-crack length is recorded at increments of 100,000 cycles, until the end of the experiment. During testing, ten crack levels were recorded for each item. The data are given in Table II, and shown graphically in Fig. 2. Similarly, we can see that nonlinearity exists in the degradation process of 2017-T4 aluminum alloy.

From the reasons given in Sections I and II, and further motivated by the above practical examples, it is natural to model both fatigue crack length and drift degradation as a stochastic process with a nonlinear path, because this model can capture...
the dynamics of the degradation process, and lead to a better understanding of the nature of the failure event.

Let $X(t)$ denote the degradation at time $t$. Then,

$$X(t) = X(0) + \int_0^t \mu(t; \theta) dt + \sigma_B B(t)$$  

(1)

where the degradation process $X(t)$ is driven by a standard Brownian motion $B(t)$ with a nonlinear drift of $\mu(t; \theta)$. In (1), $\mu(t; \theta)$ is a nonlinear function, presumed to satisfy the regularity conditions (in Ito’s sense) that guarantee a weakly unique global solution of (1) [59]. Clearly, if $\mu(t; \theta) \equiv \mu$, then (1) becomes the conventional linear drifted model in [9], [47], [53].

Considering that each item possibly experiences different sources of variations during its operation, for a degradation model to be realistic, it is more appropriate to incorporate item-to-item variability in the degradation process. As such, we treat the parameter vector $\theta$ to be a function of two parameters: $\theta = (a, b)$, where $a$ is a random effect representing between-item variation, and $b$ is a fixed effect that is common to all items. For simplicity, we assume that $\theta$ and $B(t)$ are $s$-independent, and that $a$ follows $N(\mu_0, \Sigma_a)$. The ideas of random effects and Gaussian assumptions are widely used in degradation modeling [9], [20], [21], [53], [60]. If we set $\sigma_B = 0$ in (1), then the proposed degradation model reduces to the true degradation path model adopted in conventional random-effect regression models [21], [60], [61].

We have established the degradation model using a general diffusion process. We now illustrate how to estimate the RUL based on the established model. Along the lines of the work by Aalen and Gjessing [26], and Lee and Whitmore [27], we use the concept of the FHT to define the lifetime, and then infer the RUL.

Fig. 3 illustrates the RUL modeling principle. When degradation $X(t)$ reaches a pre-set critical level $w$, the plant is declared to be non-usable, and therefore it is natural to view the event of lifetime termination as the point when the degradation process $X(t)$ crosses the threshold level $w$ for the first time. This FHT requirement may be considered to be restrictive for some cases because the degradation may go back after the first hit. However, for critical equipment, it is usually mandatory to put this FHT requirement into practice; once the observed degradation is equal to or above the set threshold level, the equipment must be stopped. From the FHT concept, the lifetime $T$ can be defined as

$$T = \inf \{ t : X(t) \geq w \mid X(0) < w \}$$  

(2)

Without loss of generality, we first consider the case $X(0) = 0$ below, and then we proceed to the case $X(t), t > 0$.

As mentioned previously, the RUL can be written as $I_T = \{t : T - t \mid T > t \}$ at current time $t$. Therefore, if $f_T(t)$ is known, then the PDF of the RUL can be formulated as

$$f_{I_T}(t) = f_T(t + I_T)/R(t)$$  

(3)

where $f_T(t + I_T)$ is the PDF of the lifetime at $t + I_T$, and $R(t)$ is the reliability function at $t$ [62]. Equation (3) is the model for a population of identical items without taking into account the individually observed $X(s), s < t$, for each item. Such $f_{I_T}(t)$, particularly when $t = 0$, is needed at the design or testing stage before the actual item is put into use in order to have an estimated designed life PDF, or to recommend a maintenance schedule for future planning purposes. As for users, once the observed degradations become available, then an individually predicted RUL is required to make a dynamic decision. In the next section, we consider situations where $I = 0$, and $I > 0$, for $X(t)$. Clearly, the key for estimating the RUL is to derive the PDF of lifetime $f_T(t)$; and thus we first focus on deriving $f_T(t)$ in the following section.

IV. LIFETIME DISTRIBUTION AND PARAMETER ESTIMATION OF THE PROPOSED DEGRADATION MODEL

A. Derivation of the Lifetime Distribution

It is difficult to derive a general analytical form of the PDF of the FHT when $\mu(t; \theta)$ is not a constant. In the following, we develop an approximate lifetime distribution in a closed form.

For simplicity, we begin without considering the random effects of the parameters in the following derivations. To derive the lifetime distribution function, we use Lemma 1 to transform the degradation process into a standard BM, which has an explicit form of the PDF of the FHT.
Lemma 1: Let there be a diffusion process \( X(t) \) with the drift \( \mu(x, t) \), and diffusion \( \sigma(x, t) \). Let \( c_1(t) \) and \( c_2(t) \) be arbitrary functions of the time. Then, iff

\[
\mu(x, t) = 4 \cdot \frac{\partial \sigma(x, t)}{\partial x} + \frac{[\sigma(x, t)]^{1/2}}{2} \times \left\{ c_1(t) + \int_z^x \frac{c_2(u)\sigma(t, u) + \partial \sigma(x, u)}{[\sigma(x, u)]^{3/2}} \, du \right\}
\]

then there exists a transformation \( \tilde{x} = \psi(t, x), \tilde{t} = \varphi(t) \) that can change the original Kolmogorov equation of the diffusion process into the Kolmogorov equation of the standard BM. This transformation can be specified as

\[
\psi(t, x) = (k_1)^{1/2} \exp \left[ -\frac{1}{2} \int_{t_0}^t c_2(\tau) \, d\tau \right] \cdot \int_z^x \frac{1}{[\sigma(y, t)]^{1/2}} \, dy
\]

\[
\varphi(t) = k_1 \int_{t_2}^t \exp \left[ -\frac{1}{2} \int_{t_0}^t c_2(u) \, du \right] \, d\tau + k_3
\]

where \( z \) is an arbitrary value in the diffusion path of \( X(t) \). Here, \( t_i \in [0, \infty), i = 0, 1, 2, \) and \( k_1, k_2, k_3 \) are arbitrary constants with the only restriction being that \( k_1 > 0 \).

The definitions of \( \mu(x, t) \) and \( \sigma(x, t) \) can be found in [36]. The proof of Lemma 1 has been well-documented by Ricciard in [40], and is thus omitted here. In this way, formulating the PDF of the FHT of \( X(t) \) crossing a constant threshold \( w \) is equivalent to calculating the PDF of the FHT of a standard BM crossing a transformed boundary using \( \tilde{x} \) and \( \tilde{t} \). Because \( t_1, t_2, k_1, k_2, k_3 \) are arbitrary constants with the only restriction that \( k_1 > 0 \), we set \( k_1 = 1 \), \( k_2 = k_3 = 0 \) in the following.

Theorem 1: For the degradation process \( X(t) \), if \( \mu(t; \theta) \) is a continuous function of time \( t \) in \([0, \infty)\), then the PDF of the FHT of \( X(t) \) crossing a critical level \( w \) can be formulated as

\[
p_{X(t)}(w, t) = p_{B(t)}(S_B(t), t) \cdot \frac{d\varphi(t)}{dt}
\]

where \( p_{B(t)}(S_B(t), t) \) is the PDF of the FHT of the standard BM \( B(t) \) crossing the corresponding threshold \( S_B(t) \). In such a case, the transformation can be written as

\[
\psi(t, x) = \frac{1}{\sigma_B} \left( x - \int_0^t \mu(t; \theta) \, dt \right), \quad \varphi(t) = t
\]

\[
S_B(t) = \frac{1}{\sigma_B} \left( w - \int_0^t \mu(t; \theta) \, dt \right)
\]

The proof is given in the Appendix.

Assumption A: The degradation process is time-homogeneous, i.e., \( \mu(t; \theta) \) is not a function of time, then \( c_2(t) \) is a constant value.

The problem under consideration can be transformed to formulate the FHT distribution of a standard BM crossing time-varying boundary \( S_B(t) \). As mentioned in the introduction, an analytical and explicit form is more desirable than a numerical approximation in reliability and RUL estimation. We therefore present the following analytical approximation. First, we give one of the lemmas presented by Durbin [63].

Lemma 2: For a Gaussian process \( W(t) \) with \( E[W(t)] = 0 \), and covariance function \( \rho(s, t) \) for \( 0 \leq s \leq t \), where \( E[I] \) is the expectation operator, assume the following.

1. The boundary function \( S(s) \) is continuous in \( 0 \leq s < t \) and is left differentiable at \( t \).
2. The covariance \( \rho(s, t) \) is positive definite, and has continuous first-order partial derivatives on the set \( \{(s, t) : 0 < s < t\} \) where appropriate left or right derivatives are taken at \( s = 0, s = t \).
3. \( \lim_{t \to r} \left[ (\partial \rho(s, t)/\partial s) - (\partial \rho(s, t)/\partial t) \right] = \lambda_r \), with \( 0 < \lambda_r < \infty \).

Then the PDF of the FHT of the process \( W(t) \) crossing the boundary \( S(t) \) can be written as

\[
p_{W(t)}(S(t), t) = b(t)h_{W(t)}(t)
\]

where \( h_{W(t)}(t) \) denotes the PDF of \( W(t) \) on boundary \( S(t) \). That is,

\[
h_{W(t)}(t) = \frac{1}{\sqrt{2\pi \sigma(t, t)}} \exp \left[ -\frac{S^2(t)}{2\sigma(t, t)} \right]
\]

and \( b(t) \) can be written as

\[
b(t) = \lim_{s \to t} (t - s)^{-1} E[W(s)W(t)]
\]

\[
\times \left[ I(s, W)(S(s) - W(s)) \right] W(t) = S(t)
\]

where \( I(s, W) \) is an indicator defined to equal 1 if the sample path of the Gaussian process of interest does not cross the boundary prior to time \( s \), and to equal 0 otherwise. Now we introduce one important assumption that will be used to prove theorem 2 as follows.

Assumption A: If the degradation process is hitting the threshold at a certain time \( t \) exactly, then the probability that such a process crossed the threshold level before time \( t \) is assumed to be negligible.

Assumption A implies that \( I(s, W) \propto 1 \) at the time of hitting the threshold, and is used for obtaining an approximate lifetime distribution. Indeed, there will be a non-zero probability that the process will cross the threshold before \( t \) in a strict sense because \( t \) is not the FHT. However, such a probability should be small because of the drift. Also, the probability of the process hitting the threshold several times should be small depending on the values of the drift and diffusion parameters. We know that degradation data are always collected at discrete time points, so we can only stop the item when we first observe that \( X(t) \) has crossed \( w \). So whether the degradation process had crossed \( w \) before is irrelevant because the exact FHT was not observed. There is also an additional advantage of using this assumption, which is that the PDF of the FHT defined here is not a strict FHT.
by definition, but rather a hitting time not far from it. There are concerns that using the FHT as the time of failure is conservative. For example, Barker and Newby [64] argued that a system could hit the critical threshold, and then return back to or below the critical level. Hence, they defined the lifetime as the last exit time from the threshold. Wang and Xu [23] recently expressed a similar idea. The same idea can also be found in the statistical control chart literature [65]. Therefore, our formulation and consideration have a practical meaning. It is difficult to theoretically assess the accuracy of the approximation, but we present several comparisons using the simulation in the Appendix to show the closeness of our approximation, and empirically validate our assumptions.

Based on Lemmas 1 and 2, Theorem 1, and Assumption A, we present the following theorem for constructing the PDF of the FHT of our established degradation process crossing a constant threshold \( w \).

**Theorem 2:** For the degradation process \( X(t) \) given by (1), if \( \mu(t; \theta) \) is a continuous function of time \( t \in [0, \infty) \), then the PDF of the FHT of \( X(t) \) crossing a constant boundary \( w \) can be approximated with an explicit form under Assumption A as

\[
p_{X(t)}(w, t) \approx \frac{1}{\sqrt{2\pi}t} \left( \frac{S_{\mu}(t)}{t} + \frac{1}{\sigma_B} \mu(t; \theta) \right) \exp \left[ -\frac{S_{\mu}^2(t)}{2t} \right]
\]

(11)

The proof is given in the Appendix.

If the degradation is monotonic, such as with a Gamma process, then a simple FHT model using \( S_{\mu}(t) = \mu(t; \theta) \) can be directly obtained. However, our process is not monotonic, and Assumption A does not imply a monotonic property. The process can hit the threshold before \( t \), or go back below the threshold after \( t \), but this probability is assumed to be small, which is consistent with the characteristics of diffusion processes. In this sense, our assumption is weaker than the monotonic assumption. The Appendix shows a comparison between the actual FHT histograms obtained from simulation and our approximated FHT distribution using Assumption A. Fig. 11 confirms that our approximation is good.

Below, we consider two special cases of \( \mu(t; \theta) \), and give the corresponding results using Theorem 2. In addition, these results will be used for the examples in Section V. Of course, other forms of \( \mu(t; \theta) \) can be selected, but we only consider these two special cases for illustrative purposes.

**Corollary 1:** Consider that the unknown parameters are fixed, and that there are no random effects among them. With the drift coefficients \( \mu(t; \theta) = \theta b(t) \), and \( \mu(t; \theta) = \theta b(t) \) corresponding to Model 1 \( (M_1) \), and Model 2 \( (M_2) \), respectively, we can obtain two different nonlinear diffusion degradation models. The PDFs of FHT under \( M_1 \) and \( M_2 \) can be formulated, respectively, from (7) and (14), as

\[
f_{T|M_1, \theta}(T \mid M_1, \theta) \approx \frac{w - a\theta b(t)(1 - b)}{\sigma_B b \sqrt{2\pi t^3}} \exp \left[ -(w - a\theta b(t))^2 / 2\sigma_B^2 t \right]
\]

(12)

\[
f_{T|M_2, \theta}(T \mid M_2, \theta) \approx \frac{w - a\theta \exp(b(t)) - b(t) \exp(b(t)) - 1}{\sigma_B b \sqrt{2\pi t^3}} \times \exp \left[ -(w - a\theta \exp(b(t)) + a)^2 / 2\sigma_B^2 t \right]
\]

(13)

where \( \theta = (\alpha, b) \).

From (12) and (13), \( f_{T|M_1, \theta}(T \mid M_1, \theta) \) can be reduced to the inverse Gaussian distribution exactly when \( b = 1 \), and both \( f_{T|M_1, \theta}(T \mid M_1, \theta) \) and \( f_{T|M_2, \theta}(T \mid M_2, \theta) \) can be reduced to the FHT distribution of the diffusion process with zero drift when \( b = 0 \), also exactly. This result is as expected because any properly developed nonlinear model should cover a linear model as a special case. However, if we use \( \Pr(T \leq t) = \Pr(X(t) \geq w) \) to obtain the FHT distribution for \( M_1 \) and \( M_2 \) as an approximation under the monotonic assumption, it can be shown without difficulty that the obtained results cannot go back to the linear and zero drift cases exactly.

In the above derivations, we assume that there is no random effect in the model parameter space. However, different items have variability in their degradation paths. This result can be interpreted as the item-to-item variability. In the current literature, considering the random effect in a parameter is a common way to characterize this variability. For simplicity, we consider that \( a \) is the random effect representing between-item variation, and that it follows a normal distribution with mean \( \mu_a \) and variance \( \sigma_a^2 \), while \( b \) is the fixed effect, and common to all items. To facilitate the derivation in the case of considering the random effect in unknown parameters, we give the following theorem.

**Theorem 3:** If \( Z \sim N(\mu, \sigma_Z^2) \), and \( w, A, B, C \in \mathbb{R} \), then

\[
E_Z \left[ (w - AZ) \cdot \exp \left( -\frac{(w - BZ)^2}{2C} \right) \right] = \sqrt{\frac{C}{B^2 \sigma^2 + C}} \times \left( w - A \frac{B^2 \sigma^2 + w + C}{B^2 \sigma^2 + C} \right) \exp \left( -\frac{(w - B\mu)^2}{2(B^2 \sigma^2 + C)} \right)
\]

(14)

The proof is given in the Appendix.

Based on Theorem 3, we can obtain the PDF of the FHT by the law of total probability. The main results are summarized in the following corollary.

**Corollary 2:** Similar to Corollary 1, if \( b \) is fixed, and \( a \sim N(\mu_a, \sigma_a^2) \), the PDFs of the FHT of \( M_1 \), and \( M_2 \) can be formulated as

\[
M_1 : f_{T|M_1}(T \mid M_1) \approx \frac{1}{\sqrt{2\pi b \sqrt{\sigma_B^2 b \tau^2 + 1} + \sigma_B^2}} \times \exp \left[ -\frac{(w - a\theta b)^2}{2(\sigma_B^2 b \tau^2 + 1) + \sigma_B^2} \right]
\]

(15)

\[
M_2 : f_{T|M_2}(T \mid M_2) \approx \frac{1}{\sqrt{2\pi b \sqrt{\sigma_B^2 b \tau^2 + 1} + \sigma_B^2}} \times \exp \left[ -\frac{(w - a\theta b)^2}{2(\sigma_B^2 b \tau^2 + 1) + \sigma_B^2} \right]
\]

(16)

with \( \gamma(t) = \exp(b(t)) - 1 \), and \( \beta(t) = \exp(b(t)) - bt \exp(b(t)) \).

**Proof:** For \( M_1 \), from Corollary 1, and using the law of total probability, we obtain

\[
f_{T|M_1}(T \mid M_1) = \frac{1}{\sigma_B \sqrt{2\pi b^3}} F_a \times \left\{ (w - a\theta b(1 - b)) \exp \left[ -(w - a\theta b)^2 / 2\sigma_B^2 t \right] \right\}
\]

(17)
Let \( A = \theta^b(1 - b), B = \theta^b, \) and \( C = \sigma_B^2 \); the result can be obtained straightforwardly using Theorem 3.

The proof for \( M_2 \) is similar, and is thus omitted. The model in [53] can be incorporated into our modeling framework when \( b = 1 \) for \( M_1 \).

To illustrate the accuracy of the proposed method in this paper, we develop an FHT simulation algorithm to generate the FHT from \( \{ X(t), t \geq 0 \} \), as summarized in the Appendix. The correspondence between the histograms generated via simulations and the PDF curves produced by our models are very close. Of course, such a simulation can be used to generate an empirical PDF of the FHT, but our method can obtain an explicit parametric form of the estimated FHT, which is desired for online realization and real-time engineering applications.

Now we proceed to our main aim, which is the estimation of the RUL at a particular point of time \( t_i \), which may be the \( i \)th monitoring point from the starting time. It is not a straightforward transformation of \( w \) by \( w = X(t_i) \), so we present another theorem as follows.

**Theorem 4:** Under the same conditions as in Corollary 2, the PDFs of the RUL of \( M_1 \) and \( M_2 \) can be formulated at time \( t_i \) with the available degradation measurement \( X(t_i) \) as

\[
\begin{align*}
\hat{f}_{L(t_i) \mid M_1}(t_i) & = \frac{1}{\sqrt{2\pi \sigma_1^2}} \left[ \frac{1}{\sigma_1^2} \left( \frac{1}{2} \right)^{w(t_i)} \right] \\
\times & \left[ \left( v_{t_i} - \eta(t_i) \right) - b(t_i) \left( t_i + v_{t_i} \right)^{-1} \\
\right. \\
\times & \left( \frac{\sigma_B^2 \eta(t_i) \left( \eta(t_i) - b(t_i) \right)}{\sigma_B^2 \eta(t_i)^2 + \sigma_B^2 \eta(t_i)} \right) \\
\times & \left[ \exp \left( - \left( \frac{w(t_i) - \eta(t_i)}{\sigma_B^2 \eta(t_i)^2 + \sigma_B^2 \eta(t_i)} \right)^2 \right) \right] \\
& \left. \right] (18)
\end{align*}
\]

with \( \eta(t_i) = (t_i + v_{t_i})^b - t_i^b, w_{t_i} = w - X(t_i); \) and

\[
\begin{align*}
\hat{f}_{L(t_i) \mid M_2}(t_i) & = \frac{1}{\sqrt{2\pi \sigma_2^2}} \left[ \frac{1}{\sigma_2^2} \left( \frac{1}{2} \right)^{w(t_i)} \right] \\
\times & \left[ \left( v_{t_i} - \beta(t_i) \right) \left( \frac{w(t_i) - \beta(t_i)}{\sigma_2^2 \beta(t_i)^2 + \sigma_2^2 \beta(t_i)} \right) \\
\times & \left[ \exp \left( - \left( \frac{w(t_i) - \beta(t_i)}{\sigma_2^2 \beta(t_i)^2 + \sigma_2^2 \beta(t_i)} \right)^2 \right) \right] \\
& \right] (19)
\end{align*}
\]

with \( \gamma(t_i) = \exp(b(t_i) + t_i) - \exp(b(t_i)), \) and \( \beta(t_i) = (1 - b(t_i)) \exp(b(t_i) + t_i) - \exp(b(t_i)). \)

The proof is given in the Appendix.

For parameter estimation, the usual MLE approach can be used to estimate the unknown parameters. In the following, we only consider \( M_1 \) and \( M_2 \) to show how to estimate the unknown parameters from the available degradation data.

**B. Parameters Estimation**

We now present a parameter estimation algorithm for the degradation model based on \( M_1 \) and \( M_2 \) to implement the derived models. We assume two cases here: one is that all items are measured at the same time, and the other is that the items are measured at different time points. The detailed presentations are summarized below.

To achieve parameter estimation, we assume that there are \( N \) tested items, and the degradation measurements of the \( n \)th item are available at time \( t_{n1}, t_{n2}, \ldots, t_{nm_n} \), where \( m_n \) denotes the available number of degradation measurements of the \( n \)th item, \( n = 1, \ldots, N \). Therefore, the sample path of the \( n \)th item at the \( j \)th point time \( t_{n_j} \) is, from (1), given by

\[
X_n(t_{n_j}) = \phi(t_{n_j})a_n + \sigma_B H(t_{n_j}),
\]

where \( j = 1, \ldots, m_n \) and \( a_n \) are \( s \)-independent and identically distributed following \( N(\mu_n, \sigma_n^2) \), as in Corollary 2.

For simplicity, we define \( \phi(t) \) as \( \phi(t) = \theta^b \), and \( \phi(t) = \exp(bt) - 1 \) for \( M_1 \) and \( M_2 \) respectively; and let \( T_n = (t_{n1}, \ldots, t_{nm_n}) \), \( T_{n_j} = \phi(t_{n_j}) \), and \( X_n = (x_n(t_{n1}), \ldots, x_n(t_{nm_n})) \), where \( (\cdot)^t \) denotes the vector transposition, and \( X \) denotes all the degradation data, consisting of \( X_n, n = 1, \ldots, N \). According to (20) and the \( s \)-independent assumption of BM, \( X_n \) follows a multivariate normal distribution with mean and variance

\[
\hat{\mu}_n = \mu_n T_n, \quad \Sigma_n = \Theta_n = \sigma_n^2 T_n \Sigma_n^t T_n, \quad (21)
\]

where

\[
Q_n = \begin{bmatrix} \mu_{n1}, \mu_{n2}, \ldots, \mu_{nm_n} \end{bmatrix} \quad \Sigma_n = \sigma_n^2 Q_n
\]

Due to the \( s \)-independence assumption of the degradation measurements of different items, the log-likelihood function over parameter set \( \Theta = (\mu_n, \sigma_n^2, \sigma_B^2, b)^t \) can be written as

\[
\ell(\theta|X) = \frac{\ln(2\pi)}{2} \sum_{n=1}^{N} m_n - \frac{1}{2} \sum_{n=1}^{N} \ln |\Sigma_n| - \frac{1}{2} \sum_{n=1}^{N} (X_n - \mu_n T_n)^t \Sigma_n^{-1} (X_n - \mu_n T_n)
\]

where

\[
|\Sigma_n| = \|\Theta_n\| \left( 1 + \sigma_n^2 T_n \Sigma_n^{-1} T_n^t \right),
\]

\[
\Sigma_n^{-1} = \Theta_n^{-1} - \frac{\sigma_n^2 T_n \Sigma_n^{-1} T_n^t}{1 + \sigma_n^2 T_n \Sigma_n^{-1} T_n^t},
\]

Taking the first partial derivative of the log-likelihood function of (23) with respect to \( \mu_n, \sigma_n \) gives

\[
\frac{\partial \ell(\theta|X)}{\partial \mu_n} = \sum_{n=1}^{N} T_n^t \Sigma_n^{-1} (X_n - \mu_n T_n)
\]

\[
\frac{\partial \ell(\theta|X)}{\partial \sigma_n} = \sum_{n=1}^{N} \frac{\sigma_n^2 T_n \Sigma_n^{-1} T_n}{1 + \sigma_n^2 T_n \Sigma_n^{-1} T_n} + \sum_{n=1}^{N} (X_n - \mu_n T_n)^t \Sigma_n^{-1} (X_n - \mu_n T_n)
\]

Case 1: Degradation measurements are available for all paths at the same time, and the number of measurements of each item
is the same; i.e., \( m_{\alpha} \) is a constant for all items, and \( t_{n,l} = t_{l,j} \) for \( n, l = 1, \ldots, N \).

Along the line of the work presented in [53], the subscript of \( T_n, \Sigma_n, \Omega_n \) in (21) – (27) can be removed. Thus, using (24) and (25), we can reduce (26) and (27) respectively to

\[ \frac{\partial \ell(\theta|X)}{\partial \mu_a} = \frac{\sum_{n=1}^{N} T_n \Omega^{-1} X_n - N \mu_a T_n \Omega^{-1} T}{1 + \sigma_a^2 T_n \Omega^{-1} T} \]  
\[ \frac{\partial \ell(\theta|X)}{\partial \sigma_a} = -\frac{N \sigma_a T_n \Omega^{-1} T}{1 + \sigma_a^2 T_n \Omega^{-1} T} \]

(28)

Then, for specific values of \( \sigma_a^2, b \), and setting these two derivatives to zero, the results of the MLE for \( \mu_a, \sigma_a^2 \) can be expressed as

\[ \hat{\mu}_a = \frac{\sum_{n=1}^{N} T_n \Omega^{-1} X_n}{N T_n \Omega^{-1} T}, \]

(30)

\[ \frac{1}{N(T_n \Omega^{-1} T)^2} \sum_{n=1}^{N} (X_n - \hat{\mu}_a T_n)' \Omega^{-1} T \Sigma_n^{-1} (X_n - \hat{\mu}_a T_n) - \frac{1}{T_n \Omega^{-1} T} \right)^{1/2}, \]

Then the profile likelihood function for \( \sigma_B, b \), in terms of estimated \( \hat{\mu}_a \) and \( \sigma_a^2 \), can be written as

\[ \ell(\sigma_B, b|X, \hat{\mu}_a, \sigma_a^2) = \frac{-N m \ln(2\pi)}{2} - \frac{N}{2} \ln |\Omega| \]

\[ -\frac{1}{2} \left\{ \sum_{n=1}^{N} X_n' \Omega^{-1} X_n - \sum_{n=1}^{N} \left( T_n \Omega^{-1} X_n \right)^2 \right\} \]

\[ -\frac{N}{2} \ln \left\{ \frac{\sum_{n=1}^{N} (T_n \Omega^{-1} X_n)^2}{N T_n \Omega^{-1} T} \right\} \]

(32)

The MLE of \( \sigma_B, b \) can be obtained by maximizing the profile log-likelihood function in (32) through a two-dimensional search. Then, substituting \( \sigma_B, b \) into (30) and (31), we can obtain the MLE for \( \mu_a, \sigma_a^2 \) accordingly.

Case 2: Degradation measurements are available for all paths at different times, with different numbers of measurements for each item.

It is clear that (27) may not have an explicit solution through setting the right-hand side of (27) to 0. Thus, for the specific values of \( \sigma_a, \sigma_B, b \), and setting the first derivatives of (23) with respect to \( \mu_a \) to zero, the result of the MLE for \( \mu_a \) can be expressed as

\[ \hat{\mu}_a = \frac{\sum_{n=1}^{N} T_n \Sigma_n^{-1} X_n}{\sum_{n=1}^{N} T_n \Sigma_n^{-1} T_n}, \]

(33)

Then, the profile log-likelihood function of \( \sigma_a, \sigma_B, b \), in terms of the estimated \( \hat{\mu}_a \), can be written as

\[ \ell(\sigma_a, \sigma_B, b|X, \hat{\mu}_a) = \frac{-N m \ln(2\pi)}{2} - \frac{N}{2} \ln |\Sigma_n| \]

\[ -\frac{1}{2} \left\{ \sum_{n=1}^{N} X_n' \Omega_n^{-1} X_n - \sum_{n=1}^{N} T_n \Sigma_n^{-1} T_n \right\} \]

(34)

The MLE of \( \sigma_a, \sigma_B, b \) can be obtained by maximizing the profile log-likelihood function in (34) through a three-dimensional search. In this paper, we use the MATLAB function “fminsearch” for this aim. The function “fminsearch” is a MATLAB function for multi-dimensional search using the simplex search method; details can be found in [67]. Now, substituting \( \sigma_a, \sigma_B, b \) into (33), we obtain the MLE for \( \mu_a \).

V. EXAMPLES OF APPLICATIONS OF THE MODELS

We illustrate three real-data examples: laser data used in [53], [61], drift degradation data from the INS, and fatigue crack data for the 2017-T4 aluminum alloy, as shown in Section III. The data analysis was performed using MATLAB.

To compare the fitting of the proposed models, the Akaike information criterion (AIC) [68], and overall mean squared errors (MSEs) of the fitted model, compared with the empirical distribution obtained directly from the data for each point in \( t_{n_1}, \ldots, t_{n_m, n} \), were both used. AIC balances the log-likelihood with the number of parameters estimated to overcome the problem of over-parameterization. The AIC is calculated as

\[ AIC = -2(\max \ell) + 2p \]

(35)

where \( p \) is the number of estimated model parameters, and \( \max \ell \) is the maximized likelihood.

MSE directly assesses the fit to the data, so it is another useful measure of goodness of fit [49], though it was acknowledged that AIC is frequently used in engineering and statistical literature to give a guideline for model selection [32], [50]. Let \( \hat{F}(t_{n,j} ; \Theta) \) denote the estimated CDF value of the lifetime at time \( t_{n,j} \) for the \( n^th \) item with estimated \( \hat{\Theta} \), and let \( \hat{F}(t_{n,j}) \) denote the empirical CDF value at time \( t_{n,j} \) for the \( n^th \) item. \( \hat{F}(t_{n,j}) \) can be estimated by the median rank method, due to Wilk and Gnanadesikan [69]. Then we have

\[ MSE = \frac{1}{N} \sum_{n=1}^{N} \frac{1}{m_n} \sum_{j=1}^{m_n} \left( \hat{F}(t_{n,j} ; \hat{\Theta}) - \hat{F}(t_{n,j}) \right)^2 \]

(36)

In both criteria, the smallest AIC and MSE values correspond to the best fitting accuracy, and thus they can serve as criteria for model selection.
A. Laser Data

First, we use the laser data in [61] to compare our methods with the work of Peng and Tseng [53], in which BM with drift was used for modeling the degradation. We chose their model as a reference for comparison primarily because it is a relatively general model that used BM with drift for degradation modeling. We refer to the model presented by Peng and Tseng [53] as $M_0$ below. A detailed data description can be found in [61]. Because the data show a clearly linear path, our objective is to show that our models are more general, and can fit this linear data set well. Similar to [53], we set the threshold to $w = 10$. Then, under $M_0$, $M_1$, and $M_2$ separately, we obtain the MLE of the unknown parameters in these models based on the method presented in Section IV-B. For comparison, we summarize the corresponding estimation results of the parameters, the log-likelihood function value (log-LF), and the mean time to failure (MTTF); and we calculate the AIC, and MSE from the fitted models (see Table III).

From Table III, we can see that our resulting log-LF and MTTF estimations are slightly different from the results of $M_0$. The model $M_0$ displays a marginally better fitting than models $M_1$ and $M_2$ in terms of the log-LF. This better performance comes from the linear nature of the laser degradation data. In addition, for $M_1$, we have $b = 1.0178$. Obviously, $b$ approaches 1 in model $M_1$. This result shows that our models can reduce to model $M_0$ if the data are linear, which validates our statement that our models are more general, and model $M_0$ is a special case of our models in the linear case. There are marginal differences among AICs, but in terms of MSE, $M_1$ and $M_2$ show a significant improvement compared with model $M_0$. It is difficult to prove the uniqueness of the MLE in this case, but we draw a contour plot with a 3-D perspective surface for $M_1$, as shown in Fig. 4, which empirically demonstrates that the profile likelihood function is convex, and the MLE is unique in this case. For the following examples, similar results can be obtained but are not shown here due to the limited space. Note that “fminsearch” is stable in all cases tested.

Correspondingly, we obtain the PDFs of the lifetime at time zero for $M_0$, $M_1$, and $M_2$, respectively, as shown in Fig. 5.

From Fig. 5, we can see that the PDFs of the estimated lifetime for $M_0$, $M_1$, and $M_2$ are almost the same, and their differences are trivial. To illustrate the usefulness of our method in RUL estimation, we select the 2nd sample in the laser data set to show the estimated RUL curves from $M_0$, $M_1$, and $M_2$ at each measuring point; see Fig. 6.

Fig. 6 shows that the differences of the PDFs of the RULs among these three models are small. This result is consistent with the results in Table III that our methods provide at least as good a fit as the linear method for this case study.

B. Drift Degradation Data of INS

The gyroscopes used in our example are very expensive, and therefore only limited tests can be performed to obtain the degradation data. Further, because of the influence of the gyroscope’s drift on the precision of a navigation system, once the observed drift value is beyond a preset threshold, it must be replaced by a new one to maintain the precision of the INS. In our experiment, the threshold is set to $0.6^\circ/\text{hour}$. In our experiment, the drift data are collected automatically, and the failure times are recorded as the times that the observed values cross over the threshold. After the experiment, the MTTF is about 21.5 hours. The drift data of the four gyroscopes are shown in Table II. Using the parameter estimation method, we obtain the corresponding estimation results of the parameters, as shown in Table IV. For comparison, we also summarize the estimated log-LF, the MTTF, the AIC, and MSE in Table IV.

---

**TABLE III**

Comparisons of Three Degradation Models with Laser Data

<table>
<thead>
<tr>
<th>$\mu_a$</th>
<th>$\sigma_a$</th>
<th>$b$</th>
<th>$\sigma_B$</th>
<th>log-LF</th>
<th>MTTF</th>
<th>AIC</th>
<th>MSE $\times 10^3$</th>
</tr>
</thead>
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<tr>
<td>$M_0$</td>
<td>0.0020366</td>
<td>0.00042148</td>
<td>—</td>
<td>0.101155</td>
<td>65.8585</td>
<td>4912.4</td>
<td>-125.7170</td>
</tr>
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<td>$M_1$</td>
<td>0.0017572</td>
<td>0.0003696</td>
<td>1.0178</td>
<td>0.010899</td>
<td>65.7802</td>
<td>4934.2</td>
<td>-123.5604</td>
</tr>
<tr>
<td>$M_2$</td>
<td>132.36</td>
<td>27.144</td>
<td>1.4935e-5</td>
<td>2.7630</td>
<td>65.6154</td>
<td>4921.4</td>
<td>-123.2308</td>
</tr>
</tbody>
</table>

---

**Fig. 4.** Profile likelihood function of $(\sigma_B, b)$ for laser data under $M_1$.

**Fig. 5.** Comparison of the PDFs of the FHT under $M_0$, $M_1$, and $M_2$ with laser data at time zero.
Table IV

Comparisons of Three Degradation Models with Drift Degradation Data of INS

<table>
<thead>
<tr>
<th></th>
<th>( \mu_z )</th>
<th>( \sigma_z )</th>
<th>( b )</th>
<th>( \sigma_b )</th>
<th>log-LF</th>
<th>MTTF</th>
<th>AIC</th>
<th>MSE ( \times 10^3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( M_0 )</td>
<td>0.055705</td>
<td>0.024954</td>
<td></td>
<td>0.20289</td>
<td>-13.898</td>
<td>13.4014</td>
<td>33.7960</td>
<td>0.3252</td>
</tr>
<tr>
<td>( M_1 )</td>
<td>2.9386e-25</td>
<td>2.7329e-25</td>
<td>18.088</td>
<td>0.0657093</td>
<td>28.376</td>
<td>23.744</td>
<td>-48.7520</td>
<td>0.0062</td>
</tr>
<tr>
<td>( M_2 )</td>
<td>9.3358e-9</td>
<td>8.6671e-9</td>
<td>0.81482</td>
<td>0.065746</td>
<td>28.540</td>
<td>22.557</td>
<td>-49.0800</td>
<td>0.0055</td>
</tr>
</tbody>
</table>

As shown in Table IV, the estimated values of \( b \) in \( M_1 \) and \( M_2 \) clearly confirm the nonlinear characteristics. As in the fatigue crack data case, Table IV shows that our models clearly outperform model \( M_0 \) in terms of the log-LF, MTTF, AIC, and MSE. Similar to the previous two studies, the PDFs of the lifetimes of \( M_0 \), \( M_1 \), and \( M_2 \) at time zero are shown in Fig. 7 for comparison.

Because the degradation paths of the gyroscopes’ drifts are nonlinear, as illustrated in Fig. 1, it is natural that models \( M_1 \) and \( M_2 \) should achieve a better fit of the PDFs of the lifetimes and the associated PDFs of RULs. Compared to \( M_0 \), the uncertainty in the estimated PDFs of the lifetimes under \( M_1 \) and \( M_2 \) is smaller, as seen in Fig. 7. Also, the location of the lifetime distribution is not far from the observed MTTF. Therefore, the validity of the proposed methods is further demonstrated.

We now use an individual item to test our obtained models, and demonstrate their results in RUL estimation at run time. We select the degradation data of the 4th item in this data set, as shown in Table I, to show the RUL estimation results under \( M_0 \), \( M_1 \), and \( M_2 \), respectively, at different measuring points based on actual observed degradation data. According to Theorem 4, the corresponding PDFs of the RULs, the estimated mean RULs, and the actual RULs for the 4th item are shown graphically in Fig. 8 for comparison.

As shown in Fig. 8, as expected, model \( M_0 \) cannot compare with the performance of models \( M_1 \) and \( M_2 \). The reason is clearly due to the nonlinearity in this case, which further demonstrates the importance of considering nonlinearity, if the degradation process is nonlinear. The prediction errors...
produced by the linear model can lead to misleading recommendations, which could result in premature replacements in the last two nonlinear examples.

C. Fatigue Crack Data of 2017-T4

The obtained data are from measured crack length propagation from four test specimens of 2017-T4 aluminum alloy (see Table II and Fig. 2). For all specimens, the crack length is measured at ten measuring points. In general, the time to cross the threshold cannot be recorded precisely. The obtained lifetime data are often interval-censored between two consecutive sampling points. Therefore, we use the data of the first time when the value is observed to be over the threshold as the approximate FHT (failure). In our experiment, the mean time to failure was about \(10^6\) cycles. Although the sample size is relatively small, the estimated results using our methods are still satisfactory in terms of the AIC and MSE, as shown below. We summarize the main estimation results of interest in Table V.

We now add one model which transforms the original data by \(\log X(t)\), which is a common way to transform nonlinear data to linear. Through this data transformation, we treat the data as linear. \(M_0\) in [53] is used. We refer to this model as \(M_0\).

As expected, parameter \(b\) in \(M_1\) is far from 1, which confirms a nonlinear degradation path. In addition, Table V shows that our models outperform models \(M_0\) and \(M_3\) in terms of the log-LF, MTTF, AIC, and MSE. Specifically, our estimations show significantly better fit in terms of both statistical goodness-of-fit measures of AIC, and MSE compared with \(M_0\) and \(M_3\). This case study demonstrates the better performance of the presented models over \(M_0\) and \(M_3\) using data transformation in the case of nonlinear degradation. We show the PDFs of lifetimes for \(M_0\), \(M_1\), \(M_2\), and \(M_3\) at time zero based on this data set in Fig. 9.

It is obvious that there are significant differences among the PDFs of the lifetimes under \(M_0\), \(M_1\), \(M_2\), and \(M_3\) with this fatigue crack data. The reason is that the fatigue crack data used in this paper display a clearly nonlinear pattern (see Fig. 2). Therefore, model \(M_0\) has a limited modeling capability in this case. This weakness can be seen from Fig. 9, because the estimated PDF under \(M_0\) of the lifetime covers a wide range so that its uncertainty is very large compared with the results of models \(M_1\) and \(M_2\). The estimated result of \(M_3\) is similar to \(M_0\), although

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TABLE V

**Comparison of Three Degradation Models with Fatigue Crack Data**

<table>
<thead>
<tr>
<th></th>
<th>(\mu_0)</th>
<th>(\sigma_0)</th>
<th>(b)</th>
<th>(\sigma_{\text{fit}})</th>
<th>Log-LF</th>
<th>MTTF (10^6) (cycles)</th>
<th>AIC</th>
<th>MSE (10^{-3})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(M_0)</td>
<td>2.6403</td>
<td>2.1055</td>
<td>—</td>
<td>3.2817</td>
<td>-63.252</td>
<td>1.1686</td>
<td>132.5040</td>
<td>276.0</td>
</tr>
<tr>
<td>(M_1)</td>
<td>7.4645e-5</td>
<td>1.4403e-5</td>
<td>12.803</td>
<td>1.762</td>
<td>-38.942</td>
<td>2.2912</td>
<td>85.8840</td>
<td>6.6</td>
</tr>
<tr>
<td>(M_2)</td>
<td>0.00014955</td>
<td>3.531e-5</td>
<td>4.4402</td>
<td>0.007788</td>
<td>-14.366</td>
<td>2.3941</td>
<td>36.7320</td>
<td>0.15795</td>
</tr>
<tr>
<td>(M_3)</td>
<td>0.76716</td>
<td>1.1135</td>
<td>—</td>
<td>1.7257</td>
<td>-42.157</td>
<td>0.8192</td>
<td>90.3140</td>
<td>390.6</td>
</tr>
</tbody>
</table>
the data are transformed to take the nonlinearity into account. The results show that such a transformation is limited in solving the nonlinear problem in this case.

Similarly, we use the degradation data of the 3rd item in this data set to show the estimated PDFs of the RULs under $M_0$, $M_1$, and $M_2$ at run time. The corresponding PDFs of the RULs, the estimated mean RULs, and the actual RULs are shown graphically in Fig. 10 for comparison.

We can see clearly from Fig. 10 that the predictions produced by models $M_1$ and $M_2$ significantly outperform those of $M_0$ and $M_3$, both in terms of the PDFs of the RULs, and the mean RULs. In Fig. 10, the observed RULs are plotted by a straight line with circle marks, while the predicted mean RULs are marked with asterisk signs. From Fig. 10, we observe that the estimated results of $M_3$ are better than $M_0$, but they are still not satisfactory. Instead, the actual RULs and predicted mean RULs from $M_1$ and $M_2$ almost overlap. This result shows the necessity of considering the nonlinearity in degradation processes if the process is or appears to be nonlinear.

VI. CONCLUSION

Motivated by our observations of real-world data sets, a diffusion process–based model was presented to characterize the dynamics and nonlinearity of degradation processes. To formulate the distribution of the RUL, we transform the problem from calculating the FHT distribution of the diffusion process crossing a constant threshold into a standard BM crossing a time-dependent boundary by using a well-known time-space transformation. Because there is no closed form for such an FHT distribution in a nonlinear case, an analytical approximation to the distribution of the FHT of the transformed process is obtained.
explicitly under a mild assumption. The unknown parameters in the degradation models are obtained using the MLE approach.

Our method differs from the existing work in two essential respects. First, we developed a degradation model using a nonlinear diffusion process, which has a nonlinear drift with time, and thus most of the current methods using a linear drift are special cases of our method. The second difference consists in providing an analytical approximation of the PDF of the FHT, thus avoiding solving the FPK equation with a constrained boundary. In the end, using three real-world data sets, we fit the developed models to the data, and demonstrated the usefulness of the proposed models and the necessity for incorporating a nonlinear structure into the degradation model if the degradation process is nonlinear.

We present several important findings. Clearly, the practical examples show that a nonlinear model is desired, and performs better for a nonlinear degradation process. However, we acknowledge that $M_0$ is easier to calculate and easier use to conduct parameter estimation. Also, $M_0$ will be the best model for the laser data set according to the AIC, but only marginally. However, in practice, nonlinearity is frequently encountered, which is largely caused by the fact that the degradation may accelerate at a late stage of the degradation process. We demonstrate that $M_0$ is inappropriate for the nonlinear cases, but our models perform very well. In this sense, our overall conclusion is that, if the degradation follows or closely approximates a linear pattern, then $M_0$ is a better model due to its simplicity; otherwise, our proposed method is a better alternative.

Although the examples demonstrated that the presented method in this paper can work better in nonlinear cases, our research is still preliminary. Specifically, our approximation approach is from a practical point of view, and more theoretical analysis is required. Also, we have not considered how to update the estimated model parameters using newly observed degradation data, which is particularly useful for online real-time control. If the degradation is latent or has measurement errors, and we only observe partial information, then a state-space model might be needed.

**APPENDIX**

A. Proof of Theorem 1

From Lemma 1, we can see that the sufficient and necessary condition for transforming the degradation process $X(t)$ to a standard BM $B(t)$ is that there exist $c_1(t)$ and $c_2(t)$, which make (4) hold. Because $\mu(x, t) = \mu(t; \theta)$, and $\sigma(x, t) = \sigma_B^2$, we directly have from (4)

$$\begin{aligned}
\mu(t; \theta) &= \frac{\sigma_B}{2} \left\{ c_1(t) + \int_0^t \frac{c_2(t)}{\sigma_B} \, dy \right\}.
\end{aligned}$$

Clearly, for (7) to hold, we can set $c_1(t) = 2\mu(t; \theta)/\sigma_B$, $c_2(t) = 0$, where $\mu(t, \theta)$ is a function of time and $\theta$. Thus, we can see that the conditions of Lemma 1 can be satisfied.

Therefore, using the transformation $\tilde{X} = X$ and $\tilde{t} = \tilde{t}$, the boundary $S_B(\tilde{t})$ for the standard BM can be calculated by

$$S_B(\tilde{t}) = \psi(\varphi^{-1}(\tilde{t}), w [\varphi^{-1}(\tilde{t})]) = \psi(t, w) = \frac{1}{\sigma_B} \left( w - \int_0^t \mu(t, \theta) \, dt \right) = S_B(t)$$

We use the result $\psi(t_0, x_0) = 0$, because $t_0 = 0$, and $x_0 = 0$, as given previously. Then we can obtain the PDF of the FHT of the diffusion $X(t)$ directly as

$$p_{X(t)}(w, t) = p_{B(t)}(S_B(t), \tilde{t}) \frac{d\varphi(t)}{dt} = p_{B(t)}(S_B(\varphi(t), \tilde{t}), \varphi(t)) \frac{d\varphi(t)}{dt} = p_{B(t)}(S_B(t), \tilde{t}) \frac{d\varphi(t)}{dt}$$

This completes the proof of Theorem 1.

B. Proof of Theorem 2

From Theorem 1 and Lemma 2, the PDF of the FHT of $X(t)$ crossing a constant threshold $w$ can be written as

$$p_{X(t)}(w, t) = p_{B(t)}(S_B(t), t) \frac{d\varphi(t)}{dt} = h(t) h_B(\tilde{t}) \frac{d\varphi(t)}{dt}$$

where $h_B(\tilde{t})$ denotes the PDF of the standard BM at boundary $S_B(t)$. Then from (8) and (9), and noting that $\rho(t, t) = t$ for a standard BM, $h(t)$, and $h_B(\tilde{t})$ can be formulated as

$$h(t) = \lim_{s \to t} t^{-s} \exp \left[ \frac{S_B^2(t)}{2t} \right]$$

Under the transformation $\tilde{t} = \varphi(t)$ in Theorem 1, we directly have $\tilde{t} = \varphi(t) = t$ from Theorem 1. Therefore there is no time-scale transformation in the process of transforming the diffusion process $X(t)$ into a standard BM. As a result, using Assumption A, we assumed that the probability of the transformed BM, $B(t)$, reaching boundary $S_B(t)$ before $t$, given that $B(t)$ crossed $S_B(t)$ at time $t$, can be neglected. As such, we can approximately have $I(s, B) \approx 1$. Then, using the property of a standard BM, and the L’Hospital rule [66], $h(t)$ can be formulated as

$$h(t) = \lim_{s \to t} t^{-s} \exp \left[ \frac{S_B^2(t)}{2t} \right]$$

This completes the proof of Theorem 2.

C. Proof of Theorem 3

Due to the limited space, we only summarize the main results below.

$$E_Z \left[ (w - AZ) \cdot \exp \left( -\frac{(w - BZ)^2}{2C} \right) \right] = wI_1 - AI_2$$
where $I_1$ and $I_2$ can be formulated separately as follows.

$$I_1 = \mathbb{E}_Z \left[ \exp \left( \frac{(w - BZ)^2}{2C} \right) \right]$$

$$= \frac{1}{\sqrt{2\pi \sigma^2}} \exp \left( \frac{-\sigma^2 w^2 + \mu C}{2\sigma^2 C} \right) \int_{-\infty}^{\infty} \exp \left( \frac{(z - \varphi)^2}{\psi} \right) dz$$

$$= \frac{1}{\sqrt{2\pi \sigma^2}} \exp \left( \frac{-\sigma^2 w^2 + \mu C}{2\sigma^2 C} \right) \cdot \exp \left( \frac{\varphi^2}{\psi} \right)$$

$$= \sqrt{\frac{C}{B^2 \sigma^2 + C}} \exp \left( \frac{- (w - B \mu)^2}{2(B^2 \sigma^2 + C)} \right)$$

with $\varphi = \sigma^2 Bw + \mu C/B^2 \sigma^2 + C, \psi = 2\sigma^2 C/B^2 \sigma^2 + C$. In a similar way, $I_2$ can be written as

$$I_2 = \mathbb{E}_Z \left[ \int_{-\infty}^{\infty} \exp \left( \frac{(z - \varphi)^2}{\psi} \right) dz \right]$$

$$= \sqrt{\frac{\psi}{2\pi \sigma^2}} \exp \left( \frac{-\sigma^2 w^2 + \mu C}{2\sigma^2 C} \cdot \frac{\varphi^2}{\psi} \right)$$

$$= \sqrt{\frac{C}{B^2 \sigma^2 + C}} \cdot \sqrt{\frac{\psi}{2\pi \sigma^2}} \exp \left( \frac{- (w - B \mu)^2}{2(B^2 \sigma^2 + C)} \right) \cdot \varphi = \varphi I_1$$

Then, the final result can be written directly as

$$E_Z \left[ (w - AZ) \cdot \exp \left( \frac{(w - BZ)^2}{2C} \right) \right]$$

$$= wI_1 - AI_2 = (w - A \varphi) I_1$$

$$= (w - A \varphi) \sqrt{\frac{C}{B^2 \sigma^2 + C}} \exp \left( \frac{- (w - B \mu)^2}{2(B^2 \sigma^2 + C)} \right)$$

This completes the proof.

D. Proof of Theorem 4

We only prove the case for $M_1$ as the proof for $M_2$ is similar. We observe $X(t_i)$ at $t_i$; then, for $t \geq t_i$, the degradation process can be written as $X(t) = X(t_i) + a(t - t_i) + \sigma_B B(t - t_i)$. In such a case, the residual $r = t_i$ corresponds to the realization of the RUL at time $t_i$ if $t$ is the FHT of $\{X(t), t \geq t_i\}$. Having this in mind, we take the transformation $\mathcal{X} = t - t_i$ with $t_i \geq 0$, and then the process $\{X(t), t \geq t_i\}$ can be transformed into $X(t) = X_t + \sigma_B B(t_i)$, where $t_i \geq 0$.

As a result, the RUL at time $t_i$ is equal to the FHT of the process $\{Y(t_i), t_i \geq 0\}$ crossing the threshold $w_{t_i} = w - X(t_i)$, where $Y(t_i) = X(t_i) - X(t_i)$, and $Y(0) = 0$. That is to say,

$$Y(t_i) = a \left( (t_i - t_i)^b - t_i^b \right) + \sigma_B B(t_i).$$

It is easy to verify that $(Y(t_i), t_i \geq 0)$ satisfies all the conditions of Theorem 1, Lemma 2, and Theorem 2; so we directly have $\mu(t_i, \theta) = \sigma_B \left( (t_i - t_i)^b - t_i^b \right)$, and $S_B(t_i) = \frac{1}{\sigma_B} \left[ w_{t_i} - a((t_i - t_i)^b - t_i^b) \right]$. Similar to the deriving process of Corollary 2, we can obtain the PDF of the RUL of $M_1$, as summarized in (18), using the results of Theorem 2 and Theorem 3 after some complicated manipulations. This completes the proof.

E. Comparison Between the Numerical Simulation and Our Explicit Result of the PDF of the FHT

To check the appropriateness of Assumption A, and the approximation accuracy of our proposed models, we develop an FHT simulation algorithm to generate the FHT from $\{X(t), t \geq 0\}$, defined as (1), to compare with the PDF of the FHT from our model. The fundamental principle of this simulation algorithm is that we can approximate $\{X(t), t \geq 0\}$ with the following so-called Euler approximation [59], [70].

$$X_{(k+1)\Delta t} = X_{k\Delta t} + \mu(k\Delta t)\Delta t + \sigma_B Y \sqrt{\Delta t}$$

where $Y \sim N(0, 1)$, and $\Delta t$ is the discretization step.

If parameter $\theta$ in the above equation is unknown, we should first use a parameter estimation algorithm to estimate $\theta$. For simplicity, but without loss of generality, we assume that the parameter set $\theta$ or its distribution is known in our simulation. Therefore, according to the definition of the FHT, we present an algorithm for simulating the FHT as follows.

Step 1: Initialize the number of sampling paths $M$, discretization step $\Delta t$, threshold $w$, and initial state $X_0$. Step 2: Launch the $m$th sampling path from the initial setting in Step 1, and let $k = 0$. Step 3: At time instant $k\Delta t$ for the $m$th sampling path, sample random numbers $\theta$ and $Y$. Step 4: Calculate $X^{(m)}_{(k+1)\Delta t}$ using an Euler approximation from $X^{(m)}_{k\Delta t}$. If $X^{(m)}_{(k+1)\Delta t} \geq w$, the FHT of the $m$th sampling path can be calculated by $T^{(m)} = (k + 1)\Delta t$, and the $m$th sampling path is terminated. As such, set $m = m + 1$, and return to Step 2. Otherwise, set $k = k + 1$, and return to Step 3 to continue the $m$th sampling path until condition $(X^{(m)}_{(k+1)\Delta t} \geq w$ is satisfied, and $T^{(m)}$ is obtained. Step 5: Repeat Steps 2–4 until $M$ FHTs are simulated, i.e., $T = \{T^{(1)}, T^{(2)}, \ldots, T^{(M)}\}$.

As an illustration, we consider a diffusion process with $dX(t) = 1.5a(t - t_i)^{1/2} dt + \sigma_B dB(t)$, where $a \sim N(1, 0.01)$. To start our simulation algorithm, we set the threshold $w = 25$, initial state $X_0 = 0$, simulation sample path number $M = 10000$, and discretization step $\Delta t = 0.01$. Because $\sigma_B$ has a dominant role for governing the process uncertainty, we give three different cases for $\sigma_B$ to compare our results with the simulated FHTs: 1) Case 1—$\sigma_B$ has a small value such as $\sigma_B = 0.05$, 2) Case 2—$\sigma_B$ has a moderate value such as $\sigma_B = 0.2$, and...
3) Case 3—$\sigma_B$ has a relatively large value such as $\sigma_B = 2$. The main results are summarized in Fig. 11.

As shown in Fig. 11, our method can generate an accurate approximation of the PDF of the FHT of the given process. Simulated FHTs are the realizations of the random FHT, so when the sample size is large, the histograms approach the true PDF of the FHT. Therefore, from the above comparisons, they provide evidence that the approximate method presented in this paper is satisfactory. This empirically validates our Assumption A.

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REFERENCES

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